The Eigenspectra of Random Graphs:
A Deterministic Approach via Edge Removal

by

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Contents

1 Abstract

Random graphs provide important models for a range of social, technological, and biological systems. The structure of these graphs is represented by an adjacency matrix A, where $A_{ij} = 1$ whenever node i and j are connected, and $A_{ij} = 0$ otherwise. The eigenvalues of these matrices are important in determining the behaviors of networked systems, from the transition to chaos in random neural networks, to epidemic thresholds for infections propagating through human populations. While the distribution of the eigenvalues of random $n \times n$ matrices as $n \to \infty$ is well understood, not much is known about the eigenvalues of random $n \times n$ matrices at finite scale. Here, we present deterministic approaches to analyze the eigenspectra and investigate the connections between regular graphs and random graphs via patterned edge removal. When viewed in a sequential manner, the effects of systematic edge removal exhibit surprising regularity. This thesis introduces the problem along with our approaches and presents numerical investigations for $n = 3, 4, 5$ and $n = 10$. We express *edge removal* as matrix multiplication, find an analytical expression for the coefficients of the characteristic polynomial in terms of the matrix entries, and use *simultaneous diagonalization* in order to study this problem. Furthermore, we prove that the eigenspectrum of the complete graph with one edge removed is the same regardless of *which* edge we choose to remove. Finally, our numerical investigation suggests that when two edges are removed, there are four unique eigenspectra formed, including the trivial eigenspectrum, regardless of n.

2 Introduction

In this section, we will present introductory concepts from graph theory and linear algebra used throughout this thesis.

2.1 Graph Theory

A *graph* is a mathematical object containing n nodes, or *vertices*, and with connections, or *edges*, between some pair of nodes. We can assign each node some arbitrary label $i = 1, 2, \ldots, n$ [2]. Furthermore, a graph G has *vertex set* $V(G)$ and *edge set* $E(G)$, where an edge from node i to node j is denoted by the 2-tuple (i, j). The idea of *direction* of an edge is reflected in the order in which we write the edge. A graph G is *directed* if every $(i, j) \in E(G)$ is an *ordered* 2-tuple, and G is *undirected* otherwise. A graph is *simple* if there are no self-loops, that is, edges of the form (i, i), and there is at most 1 edge between any pair of nodes.

Suppose G is a graph with the vertex set $V(G)$ and the edge set $E(G)$. Moreover, $|V(G)| = n$. Let us define the matrix $A \in M_{n\times n}(F)$ such that for $i, j \in V(G)$,

$$A_{ij} = \begin{cases} 1 & \text{if } (i,j) \in E(G) \\ 0 & \text{otherwise} \end{cases}$$

With this definition, A is the *adjacency matrix* representation of G. We also note that A is symmetric if G is an undirected graph. Furthermore, if $\lambda_1, \lambda_2, \ldots, \lambda_k \in \mathbb{F}$ are the eigenvalues of A with the corresponding eigenvectors $v_1, v_2, \ldots, v_k \in \mathbb{F}^n$, then they are the eigenvalues and eigenvectors of G.

We can characterize different graphs with the definitions above. For example, the *null graph* is the graph with $V(G) = E(G) = \emptyset$. We are particularly interested in the *complete graph*, K_n, on n nodes. This is the graph such that for every pair of nodes i, j and $i \neq j$, there exists, without loss of generality, the edge (i, j). Thus the matrix representation of K_n is

$$[K_n]_{ij} = \begin{cases} 1 & \text{if } i \neq j \\ 0 & \text{otherwise} \end{cases}$$

For example, for $n = 3$, then

$$[K_3] = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

We can capture everything about a graph with its matrix representation, and thus it is possible to use results from linear algebra in order to analyze graphs further.

2.2 Linear Algebra

To begin, we introduce a classic result from linear algebra. The matrices A, B are *similar* if there exists P such that $A = P^{-1}BP$. Notice that if λ is an eigenvalue of A, then

$$A\vec{v} = \lambda\vec{v} \iff P^{-1}BP\vec{v} = \lambda\vec{v} \iff BP\vec{v} = \lambda P\vec{v} \iff B\vec{v} = \lambda\vec{v}$$

Thus A, B share the same eigenvalues.

The $n \times n$ matrix Q is *orthogonal* if $Q^T = Q^{-1}$, where Q^T is the transpose of Q. The $n \times n$ complex matrix U is *unitary* if $U^* = U^{-1}$, where U^* is the conjugate transpose of U. A matrix C is *circulant* if it is of the form

$$C = \begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-1} \\ c_{n-1} & c_0 & c_1 & & \vdots \\ \vdots & c_{n-1} & c_0 & \ddots & \\ & \ddots & \ddots & & \\ c_1 & \cdots & & c_{n-1} & c_0 \end{pmatrix} = \operatorname{circ} \underbrace{\begin{pmatrix} c_0 & c_1 & \dots & c_{n-1} \end{pmatrix}}_{\vec{c}}$$

That is, given a generating row vector $\vec{c}$, the first row of C is $\vec{c}$, the second row of C is $\vec{c}$ shifted 1 entry to the right, the third row of C is $\vec{c}$ shifted 2 entries, etc. Each row of C is a cyclic shift of the previous row [3]. Furthermore, the eigenspectrum of C can be expressed analytically as

$$\lambda_m = \sum_{k=0}^{n-1} c_k \omega^k$$

where $\omega = e^{-2\pi i m/n}$, the n^{th} complex root of unity. The corresponding eigenvectors are given by

$$y^{(m)} = \frac{1}{\sqrt{n}} \begin{pmatrix} 1 & \omega^m & \omega^{2m} & \dots & \omega^{(n-1)m} \end{pmatrix}$$

An immediate corollary of this result is that there exist U and U^{-1} unitary matrices such that

$$U^{-1} C U = D$$

where D is the diagonal matrix with the eigenspectrum of C on its diagonal [3].

Another relevant result from linear algebra that we want to introduce is *simultaneous diagonalization.* Let $(V, \langle \cdot, \cdot \rangle)$ be a finite-dimensional complex inner product space with $\langle x, y \rangle = x^* y$, for all $x, y \in V$, where x^* is the conjugate transpose of x. Recall that a linear transformation $T : V \to V$ is *Hermitian* (self-adjoint) if for all $x, y \in V$, $\langle Tx, y \rangle = \langle x, Ty \rangle$. Let A be the transformation matrix of T with respect to some basis $\mathcal{B}$ of V, then $A = A^*$. Moreover, suppose $\mathcal{A}$ is a collection of transformation matrices of commuting Hermitian linear transformations on V, that is, $AB = BA$ for $A, B \in \mathcal{A}$. Then V has a basis consisting of eigenvectors for all $A \in \mathcal{A}$.

3 Deterministic Approach

3.1 Statement of Problem

Currently we are equipped with various probabilistic tools to investigate random $n \times n$ matrices as $n \to \infty$, but these tools are not applicable for random matrices at finite scale. In this thesis, we attempt to develop a deterministic way to approach random $n \times n$ matrices. First, we fix the number of nodes, let that number be n, and consider the adjacent matrix of the complete graph on n nodes, which is defined by

$$[K_n] = \text{circ}\underbrace{\begin{pmatrix}0 & 1 & 1 & \dots 1\end{pmatrix}}_{\text{n}}$$

Note that $[K_n]$ is a circulant matrix, and thus its eigenvalues are completely determined.

Suppose $A_{n\times n}$ is the adjacency matrix of some simple undirected graph G on n nodes, then it can be argued that there exists a sequence $S = (E_1, E_2, \dots, E_k)$ for some $1 \leq k \leq |E(G)|$, such that E_i is an $n \times n$ matrix and has exactly 2 non-zero entries and

$$A = [K_n] - \sum_{i=1}^{k} E_i$$

Similarly, for the directed case, there exists a sequence $S = (E_1, E_2, \dots, E_k)$ for some $1 \leq k \leq |E(G)|$ such that E_i is an $n \times n$ matrix and has exactly 1 non-zero entry and the expression of A above holds. Furthermore, this thesis claims that S is determined by the complement of A. This concept is best explained with an example. Let

$$A = \begin{pmatrix}0 & 0 & 1\\ 0 & 0 & 1\\ 1 & 1 & 0\end{pmatrix}$$

then in the undirected case, let

$$E_1 = \begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$$

and thus $A = K_3 - E_1$. In addition, in the directed case, let

$$E_1 = \begin{pmatrix}0 & 1 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix} \qquad \text{and} \qquad E_2 = \begin{pmatrix}0 & 0 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$$

then $A = K_3 - E_1 - E_2$. We note that the E_i's are the decomposition of the complement matrix of A, say A^C, which is defined by

$$A^C = J - I_n - A = K_n - A$$

where J is the all-ones $n\times n$ matrix and I_n is the $n\times n$ identity matrix. This is a formal representation of what we will refer to in this thesis as *edge removal*. In general, edge removal can be thought of as letting an entry of the complete matrix be 0, that is, (i, j) is removed if and only if we let $[K_n]_{ij} = 0$. Suppose we have a simple undirected or directed graph G on n nodes, then we can reach G by removing a certain sequence of edges from K_n, the complete graph on n nodes, which motivates our investigation on the effect of edge removal on the complete graph and its eigenspectrum. This is a novel approach in studying random matrices, as this allows us to make conclusions without letting $n \to \infty$.

3.2 Edge Removal As Matrix Multiplication

As introduced above, *edge removal* can be expressed as matrix subtraction. However, it is natural to seek a more well-studied representation of *edge removal*. In this section, we look to express *edge removal* as matrix multiplication. Notice that $[K_n]$ is an invertible matrix, and therefore the set of the columns (or rows) of $[K_n]$,

$$\beta = \left\{ \vec{v}_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ \vdots \\ 1 \\ 1 \end{pmatrix}, \dots, \vec{v}_n = \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \\ 0 \end{pmatrix} \right\}$$

forms a basis for $\mathbb{R}^n$. We prove the following lemma.

Lemma: For all $A \in M_{nn}(\mathbb{R})$, there exist vectors $\vec{c}_i$ of length n (for $i \in \{1, 2, \dots, n\}$), such that for $k \in \{1, 2, \dots, n\}$,

$$A_{i,k} = \vec{c}_i \cdot \vec{v}_k \text{ where } \cdot \text{ is the inner product}$$

Proof. Fix an adjacency matrix $A \in M_{nn}(\mathbb{R})$. Let the set of rows of A be $\alpha = \{\vec{r}_1, \vec{r}_2, \dots, \vec{r}_n\}$, then there exist the linear combinations

$$(\vec{r}_i)^T = \sum_{j=1}^{n} c_{i,j} \vec{v}_j \tag{1}$$

where $i \in \{1, 2, \dots, n\}$ and $c_{i,j} \in \mathbb{R}$. Define $\vec{c}_i \in \mathbb{R}^n$ such that

$$\vec{c}_i \coloneqq (c_{i,j})_{j=1}^{n} \tag{2}$$

It follows that

$$A_{i,k} = (\vec{r}_i)_k^T \text{ where } i, k \in \{1, 2, \dots, n\} \tag{3}$$

$$= \sum_{j=1}^{n} c_{i,j} (\vec{v}_j)_k \tag{4}$$

$$= c_{i,1}(\vec{v}_1)_k + c_{i,2}(\vec{v}_2)_k + \cdots + c_{i,n}(\vec{v}_n)_k \tag{5}$$

The vector $[(\vec{v}_1)_k, (\vec{v}_2)_k, \dots, (\vec{v}_n)_k]$, such that $l \in \{1, 2, \dots, n\}$ is the k^{th} row of K_n, which is also $\vec{v}_k$ since the adjacency matrix of K_n is symmetric. Thus we have

$$A_{i,k} = \vec{c}_i \cdot \vec{v}_k \tag{6}$$

where $\cdot$ is the usual dot product. □

Theorem: For any $A \in M_{nn}(\mathbb{R})$, there exists $T \in M_{nn}(\mathbb{R})$ such that $TK_n = A$.

Proof. We construct the transformation matrix $T \in M_{nn}(\mathbb{R})$ such that

$$T_{i,j} = c_{i,j}$$

Now, we show that $T(K_n) = A$. Let $\vec{v} = \big(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\big)$, then

$$\begin{aligned} T(K_n) &= \begin{pmatrix} c_{1,1} & c_{1,2} & \cdots & c_{1,n-1} & c_{1,n} \\ c_{2,1} & c_{2,2} & \cdots & c_{2,n-1} & c_{2,n} \\ & & \vdots & & \\ c_{n,1} & c_{n,2} & \cdots & c_{n,n-1} & c_{n,n} \end{pmatrix} \vec{v} \\ &= \begin{pmatrix} \vec{c}_1 \\ \vec{c}_2 \\ \vdots \\ \vec{c}_n \end{pmatrix} \big(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\big) \\ T(K_n)_{i,j} &= \vec{c}_i \cdot \vec{v}_j \text{ where } \cdot \text{ is the inner product} \\ &= A_{i,j} \text{ (follows from lemma)} \end{aligned}$$

□

We have shown that given an adjacency matrix A of a graph G on n nodes, there exists a transformation matrix T such that $A = T[K_n]$. Furthermore, we have proved the analytical form of T. This is true in particular for A, the adjacency matrix for the complete graph with some edges removed. A downside of this transformation approach is that this T cannot be used repeatedly on K_n, as the columns of $A = T[K_n]$ are not always linearly independent. Furthermore, an extension of this approach is to study the eigenspectrum of T. If we found that K_n and T share the same eigenvectors, then a direct corollary would be

$$A\vec{v} = (T[K_n])\vec{v} = T([K_n])\vec{v} = T\lambda\vec{v} = \lambda(T\vec{v}) = \lambda\omega\vec{v}$$

where λ is an eigenvalue of $[K_n]$ and ω is an eigenvalue of T. It would follow that $\lambda\omega$ is an eigenvalue of A. Thus we would be able to express the eigenspectrum of A in terms of products of the eigenvalues of both $[K_n]$ and T. However, it can be the case that there is no general expression of the eigenspectrum of T, and we wish to further investigate this topic in future work.

3.3 Coefficients Of The Characteristic Polynomial

Since the eigenvalues of A are the roots of its characteristic polynomial, an alternative approach is to study its characteristic polynomial. The eigenvalues can then be found using existing efficient numerical root-finding methods. As mentioned before, *edge removal* from the complete matrix on n nodes can be seen as a change in the entries of $[K_n]$, and thus it is important to express the characteristic polynomial of A in terms of its entries. This section employs the results from *The Coefficients Of The Characteristic Polynomial In Terms Of The Eigenvalues And The Elements Of An $n \times n$ Matrix* by *Dr. Bernard P. Brooks.* Let us express the characteristic polynomial of the matrix A as

$$\begin{aligned} C_A(x) &= (-1)^n \det[A - xI] \\ &= \sum_{k=0}^{n} c_k x^k \end{aligned}$$

where c_k are the coefficients of the x^k terms. It is known that c_k can be computed by summing the $\binom{n}{k}$ determinants of the matrices created by replacing k of the diagonal elements of matrix A with -1 and the remaining elements in those corresponding rows and columns with 0 [1]. For example, consider

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 & 1 \\ 1 & 0 & 5 & 1 & 4 \\ 1 & 2 & 3 & 4 & 5 \\ 4 & 5 & 4 & 1 & 2 \\ 1 & 3 & 4 & 5 & 2 \end{pmatrix}$$

Thus there exists a direct connection between the coefficients of the characteristic polynomial of adjacency matrices after edge removal. First, we notice that $\text{trace}(K_n) = 0$, and it remains 0 even after edge removal. In addition, it is known that the coefficient of the x^{n-1} term is the trace of the matrix, and thus is always 0. We can also say that the sum of the eigenvalues is always 0. Let $c = \begin{pmatrix} c_n & c_{n-1} & \dots c_1 & c_0 \end{pmatrix}$ be the vector of the coefficients of the characteristic polynomial of $[K_n]$. Given that edge (i, j) is removed from K_n, resulting in the adjacency matrix A, we are interested in $c' = \begin{pmatrix} c'_n & c'_{n-1} & \dots c'_1 & c'_0 \end{pmatrix}$, the coefficient vector of A. In particular, we are interested in

$$c - c' = (c_k - c'_k)_{k=0}^{n}$$

For fixed k, we can express $c_k - c'_k$ as sum of determinants [1]. Thus $c_k - c'_k$ is the sum of $\binom{n-2}{k}$ determinants. For example, consider

$$[K_3] = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

with $C_{K_3} = x^3 + c_2 x^2 + c_1 x + c_0$. Since K_n is a circulant matrix, we can find an analytical expression for the characteristic polynomial for K_n. Suppose edges $(1, 2)$ and $(2, 1)$ are removed, then let

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

and $C_A(x) = x^3 + c'_2 x^2 + c'_1 x + c'_0$. As stated before, $c_2 = 0$, and $c_0 = \det(A)$. Then we can write $c_1 - c'_1$ as

$$(-1)^3 \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{vmatrix} - (-1)^3 \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{vmatrix}$$

Given an adjacency matrix, we can compute the change in the coefficients of the characteristic polynomials from the characteristics polynomial of $[K_n]$. We hope to develop an analytical expression of the coefficients of $[K_n]$ is k edges removed in terms of sums of determinants in future work.

3.4 Simultaneous Diagonalization

In this section, we pose the *edge removal* problem through the lens of matrix theory. Consider *simple* and *undirected* graphs, and thus the adjacency matrices we are concerned with are symmetric, binary and have only 0 on the diagonal. Let's introduce some notation. Let J be the $n \times n$ matrix with all 1 entries. In addition, we will denote the eigenspectrum of the adjacency matrix A by $\lambda(A)$. Given adjacency matrix A of the graph G on n nodes, recall that the complement of A is defined by

$$A^c = J - I - A = K_n - A$$

where K_n is the adjacency matrix of the complete graph on n nodes, and thus we have that $A = K_n - A^c$. We can decompose A^c into a sum of matrices $E_1, \ldots, E_k$ for $1 \leq k \leq |E(G)|$ such that E_k is symmetric, and has exactly 2 non-zero entries. For example, when $n = 3$,

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

The adjacency matrix K_n of the complete graph is circulant, and thus we can ultilize its well-determined properties. Its spectrum can be expressed analytically as

$$\lambda_j = \sum_{i=1}^{n-1} \omega^{kj} \text{ for } j = 0, \ldots, n-1$$

where $\omega = e^{\frac{2\pi i}{n}}$. Now, the next step is to investigate how $\lambda(K_n)$ changes as we subtract the E_k's. Note that this is analogous to removing edges from the complete graph on n nodes. Here, we have a discrete perturbation when changing the entries of K_n directly, which diverges from traditional perturbation theory. The problem can now be formulated as: For $A, B, C \in M_{n\times n}(\mathbb{R})$, if $C = A + B$, and $\lambda(A), \lambda(B)$ are known, what can we say about $\lambda(C)$? If A, B are arbitrary matrices, this is a difficult question to ask, but we are concerned with a very specific types of matrices.

Suppose A, B are Hermitian complex matrices and $AB = BA$. Suppose $A, B \in \mathcal{A}$, and consider $C = A + B$. By the *simultaneous diagonalization* result, there exist P and P^{-1} such that

$$\begin{aligned} P^{-1}CP &= P^{-1}(A + B)P \\ &= P^{-1}AP + P^{-1}BP \\ &= D_A + D_B = D_C \end{aligned}$$

where D_A, D_B, D_C are diagonal matrices with the eigenvalues of A, B, C, respectively, on the diagonal. Here, we are able to express $\lambda(C)$ as a sum of $\lambda(A)$ and $\lambda(B)$. Studying the commutator of any pair of the E_i's is the logical next step.

Because K_n and E_k's are real and symmetric matrices, they form a Hermitian set of matrices. It is left to check if they commute. For example, for $n = 3$,

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

We can see that they do not commute, but observe that $K_n E_k = (E_k K_n)^T$. This means that we cannot apply the theorem directly, and the question remains, what can we say about $\lambda(K_n + E_k)$, given such a precise relationship between K_n and E_k. Generally, there is no immediate result about the eigenspectrum of a sum of matrices, except for some bounds on their eigenspectrum.

4 Numerical Investigation

In our numerical investigation, we investigate directed graphs of size $n = 3, 4, 5$ and $n = 10$.

4.1 Methodology

First, we fix a sequence of edges $\mathcal{R}$ to remove from K_n, so that we get to the zero $n \times n$ matrix. For example, for $n = 3$, suppose

$$\mathcal{R} = \{(1,2),(2,3),(3,2),(2,1),(1,3),(3,1)\}$$

Then $E(K_3) \setminus \mathcal{R} = \emptyset$. At each removal step, we iterate over all entries of $[K_n]$ with some edges removed, temporarily remove that entry, and record the eigenspectrum of the resulting matrix. Going back to our example, at the first step of removal, we remove edge $(1,2)$ from K_3, that is,

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

Next, we record the eigenspectrum of the following matrices

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & \underline{\mathbf{0}} \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ \underline{\mathbf{0}} & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & \underline{\mathbf{0}} \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ \underline{\mathbf{0}} & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & \underline{\mathbf{0}} & 0 \end{pmatrix}$$

We can then define the $3 \times 3 \times 3$ array M as follows

$$\begin{aligned} M_{ij} \coloneqq \text{ the eigenspectrum (expressed in a vector of length 3) of } K_3 \\ \text{with edges } (1,2) \text{ and } (i,j) \text{ removed} \end{aligned}$$

We repeat this process for K_3 with edges $(1,2)$ and $(2,3)$ removed, etc. The entries of the array M can then be colour coded, depending on the entry removed along with the predetermined removal path. In particular, the entries will be of the same colour if removing those entries at the removal step produces the same eigenspectrum.

4.2 Examples

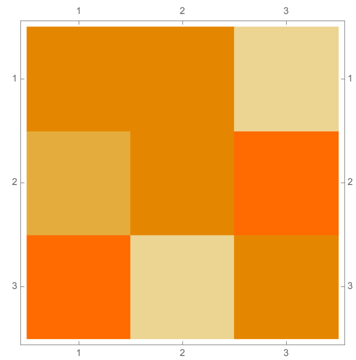
Suppose the removal sequence

$$\mathcal{S} = \big\{(1,2),(3,2),(3,1),(1,3),(2,3),(2,1)\big\}$$

1. Remove $(1,2)$, then we have

$$A_1 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

and M is



From this plot we can see that there are 4 unique eigenspectra produced. In particular, the adjacency matrices in the following sets produce the same eigenspectra.

$$C_1 = \left\{ \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \right\}$$

$$C_2 = \left\{ \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \right\}$$

$$C_3 = \left\{ \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} \right\}$$

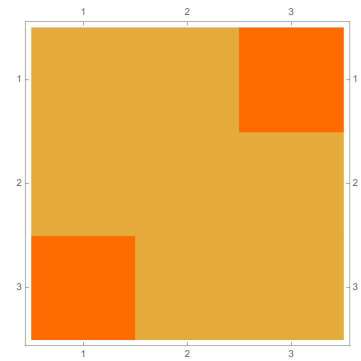
$$C_4 = \left\{ \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \right\}$$

Notice that C_1 is the eigenspectrum of the matrix with no change. C_2 is the matrix with the edge opposite from $(1,2)$, i.e. $(2,1)$, removed. C_3 is the matrices with edges with the heads or tails being 1 or 2. Finally, C_4 is the rest of the variations of A.

2. Remove $(1,2)$ and $(3,2)$, then we have

$$A_2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

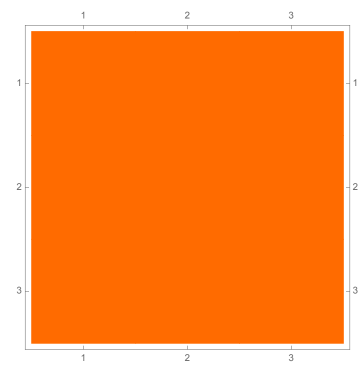
and M is



3. Remove $(1,2)$, $(3,2)$, and $(3,1)$, then we have

$$A_3 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

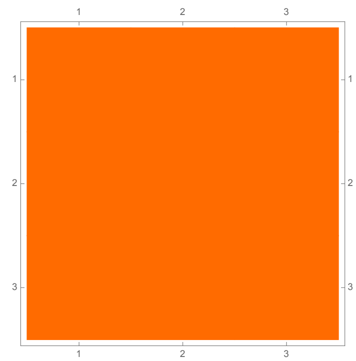
and M is



4. Remove $(1,2)$, $(3,2)$, $(3,1)$, and $(1,3)$, then we have

$$A_4 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

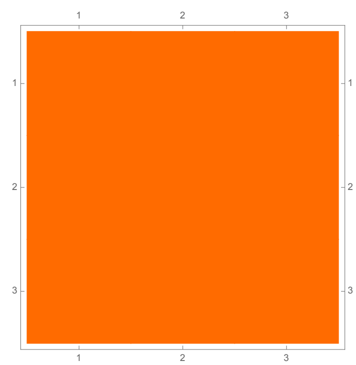
and M is



5. Remove $(1,2)$, $(3,2)$,$(3,1)$, $(1,3)$, and $(2,3)$, then we have

$$A_5 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

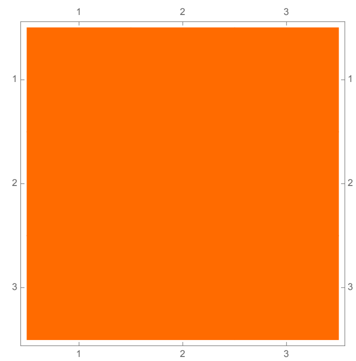
and M is



6. Remove $(1,2)$, $(3,2)$,$(3,1)$, $(1,3)$, $(2,3)$, and $(2,1)$, then we have

$$A_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and M is



4.3 The Number of Unique Eigenspectra - "Clusters"

We employ the *Monte Carlo method* to investigate the number of unique eigenspectra, or *clusters*, produced at each step of the edge-removal process described above.

4.3.1 n=3

The number of possible edges we can remove is 6, thus there are 6! possible sequences of edges that we can remove. In this experiment, 1000 sequences of edges were randomly chosen, and we recorded the average number of clusters formed.

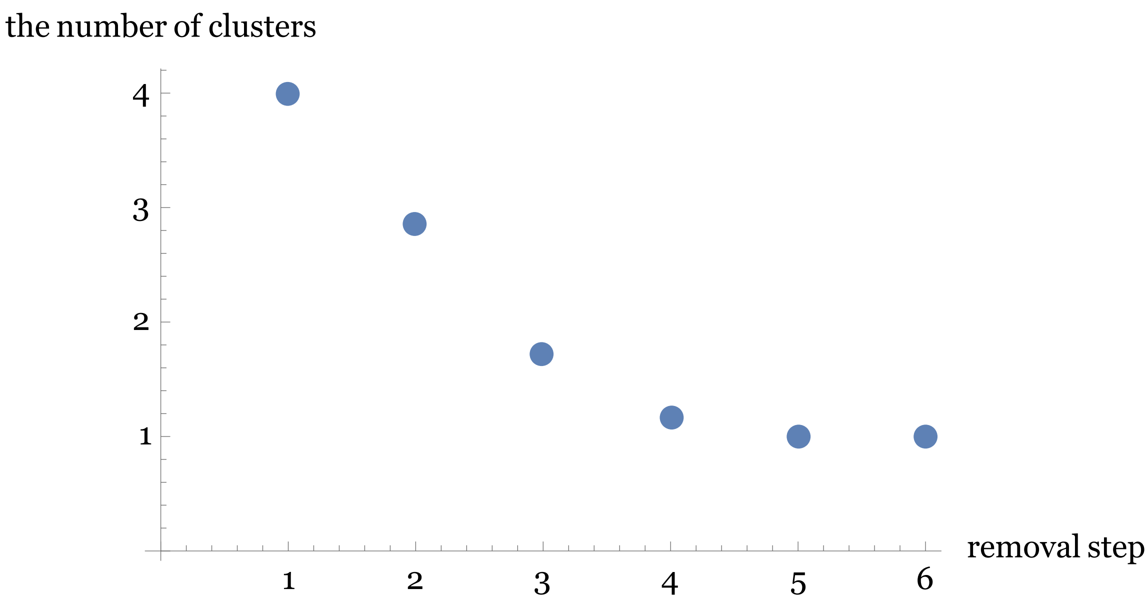


Figure 1: The average number of clusters at each removal step for $n = 3$

4.3.2 n=4

The number of possible sequences of edges to remove is 12!. This experiment is repeated 1500 times.

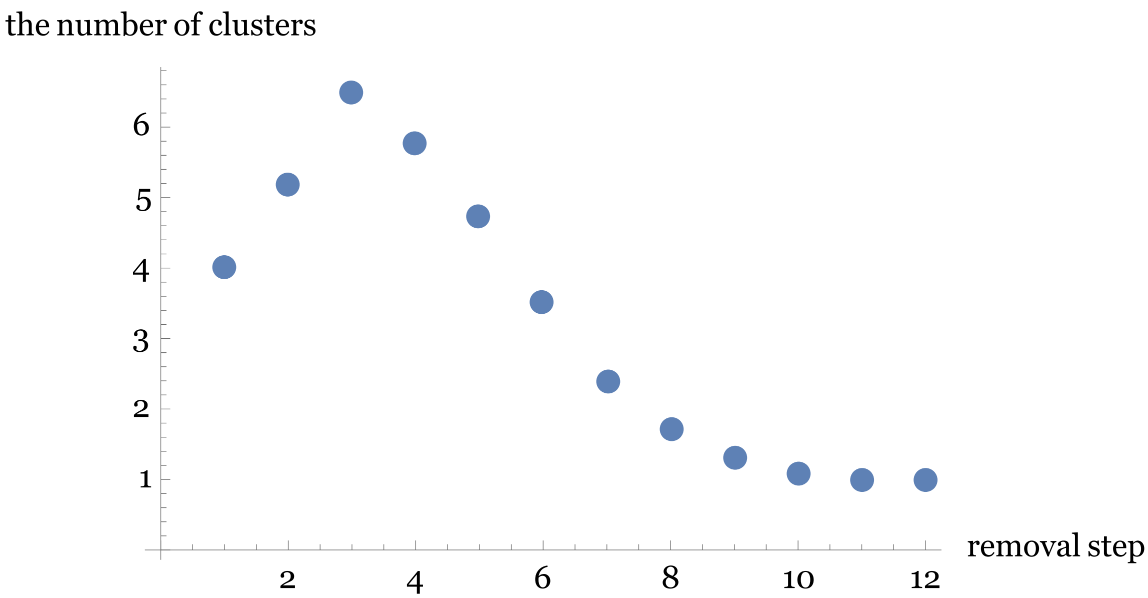


Figure 2: The average number of clusters at each removal step for $n = 4$

4.3.3 n=5

The number of possible sequences of edges to remove is 20!. This experiment is repeated 1500 times.

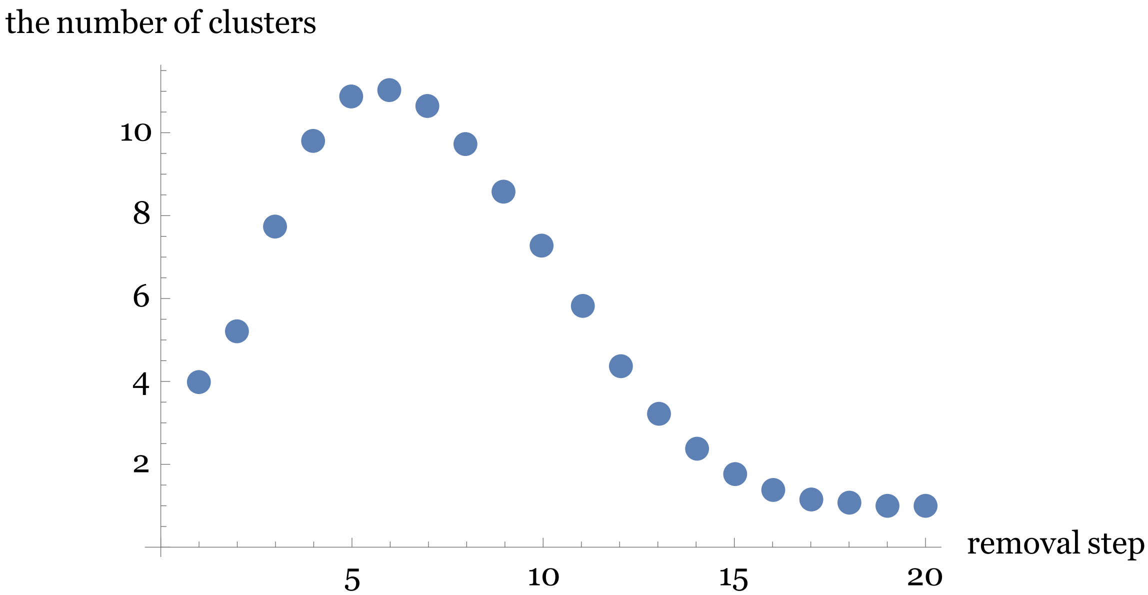


Figure 3: The average number of clusters at each removal step for $n = 5$

4.3.4 n=10

The number of possible sequences of edges to remove is 90!. This experiment was repeated 1000 times.

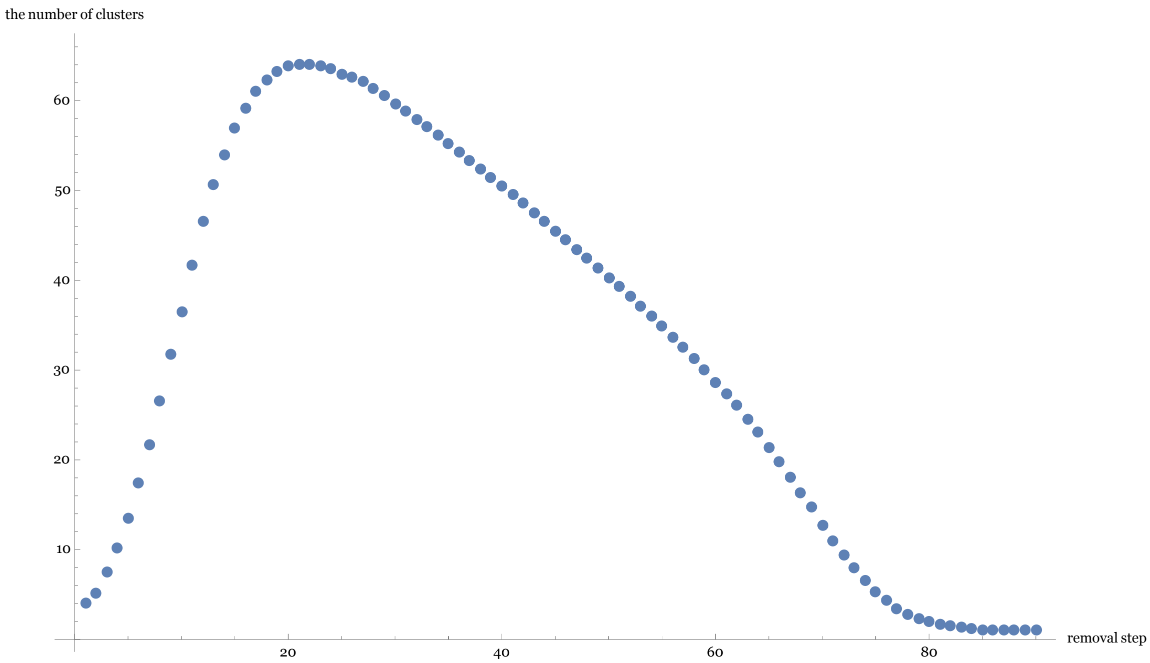


Figure 4: The average number of clusters at each removal step for $n = 10$

5 Discussion

In this section, we will discuss the directed case, similar to the numerical investigation above. First, the eigenspectrum of the complete matrix with the first edge removed is independent from *which* edge was removed. Intuitively, this is because the labeling of the nodes is completely arbitrary. To formalize this idea, let us prove the following claim.

Claim: Let G_1, G_2 be graphs on the same n nodes such that

$$(G_1)_{rc} = \begin{cases} 0 & r = c \text{ or } r = i, c = j \\ 1 & \text{otherwise} \end{cases} \quad \text{and} \quad (G_2)_{rc} = \begin{cases} 0 & r = c \text{ or } r = k, c = l \\ 1 & \text{otherwise} \end{cases}$$

In other words, G_1 is the complete graph with edge (i, j) removed and G_2 is the complete graph with edge (k, l) removed. Then G_1 and G_2 are *isomorphic*. We denote this by $G_1 \cong G_2$.

Proof. Recall that the graphs G_1, G_2 are *isomorphic* if there exists a bijection, say φ, from $V(G_1)$ to $V(G_2)$ such that $(i, j) \in E(G_1)$ if and only if $(\varphi(i), \varphi(j)) \in E(G_2)$. In terms of matrix representation, the graphs G_1, G_2 with adjacency matrices A_1, A_2 are *isomorphic* if there exists an orthogonal $n \times n$ matrix Q such that

$$A_1 = Q^T A_2 Q$$

For $n = 2$, the claim is trivial, as there are only 2 possible edges to remove, and we can let $\varphi(1) = 2$ and $\varphi(2) = 1$ be the required bijection. For $n \geq 3$, let φ be such that for $x \in V(G_1)$

$$\varphi(x) = \begin{cases} k & \text{if } x = i \\ l & \text{if } x = j \\ i & \text{if } x = k \\ j & \text{if } x = l \\ x & \text{otherwise} \end{cases}$$

Then φ is a bijection. In fact, its inverse is itself. Intuitively, φ can be thought of the mapping from the vertex set of G_1 to the vertex set of G_2 such that if the node is not i or j, φ maps that node to itself. Otherwise, if the node is i, then φ maps it to k. Analogously, φ maps j to k. If we apply φ twice, $i \mapsto k \mapsto i$ and $j \mapsto l \mapsto j$, while the other nodes stay the same. Thus $\varphi \circ \varphi = id$. Furthermore, it suffices to show the inverse and contrapositive of the statement "$(i, j) \in E(G_1)$ if and only if $(\varphi(i), \varphi(j)) \in E(G_2)$ ". Since only (i, j) and (k, l) are not in $E(G_1)$ and $E(G_2)$, respectively, we have that

$$(i, j) \notin E(G_1) \iff (\varphi(i), \varphi(j)) = (k, l) \notin E(G_2)$$

we have proved our claim. Thus $G_1 \cong G_2$. $\square$

From the matrix representation of isomorphic graphs, we can see that since Q is orthogonal, $Q^T = Q^{-1}$, and thus A_1 and A_2 are *similar* matrices. It follows that they share the same eigenvalues. Thus we can conclude that the eigenspectrum resulting from removing the first edge from the complete graph is the same regardless of which edge we choose to remove. The eigenspectrum gets more complicated when 2 edges are removed, that is, removal step 1 in our numerical investigation. The plots above suggest that there are 4 clusters of eigenspectra formed at the removal step 1, regardless of n. At the last removal step, there is 1 cluster formed, which is the eigenspectrum of the zero matrix. It is important to note that if $G_1 \cong G_2$, then they have the same eigenspectrum, however, the converse is not always true. There are *cospectral* graphs but not isomorphic, that is, they share the same eigenspectrum, but are not isomorphic. The number of clusters formed can be seen to follow a particular analytical function, which motivates our belief that this is a restriction of a continuous, analytical function. We hope to further develop our approaches mentioned in previous sections, classify the eigenspectra up to isomorphism, and seek an explanation for the numerical results found, in particular, the potentially unique maximum number of the average number of clusters.

6 Appendices

6.1 Appendix A: Mathematica Notebook

[Redacted as the author's name can be found on Github]

References

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[3] Robert M Gray. "Toeplitz and circulant matrices: A review". In: (2006).